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LEAST-INDEX RESOLUTION OF
DEGENERACY IN LINEAR COMPLEMENTARITY PROBLEMS,

by

10 Yow-Yieh Chang

9 TECHNICAL REPORT 79-14

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LEAST-INDEX RESOLUTION OF DEGENERACY IN LINEAR COMPLEMENTARITY PROBLEMS

by

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ABSTRACT

This study centers on the circling phenomenon associated with degeneracy in linear complementarity problems and presents an easily implemented technique for resolving it. With certain exceptions, the device is to use the least-index for selecting the variable to leave the basic set.

The results of this report pertain only to linear complementarity problems involving P-matrices or positive semi-definite matrices. With this restriction, it is shown that inclusion of the least-index pivot selection rule insures finiteness for the principal pivoting method of Dantzig and Cottle, Lemke's algorithm, and Cottle's parametric principal pivoting method. It is shown that for circling to occur in the principal pivoting method, the matrix must have order at least four, and for Lemke's algorithm it must be at least three. Examples are given showing that these bounds are sharp. Finally, Murty's version of Bard's method is extended from P-matrices to the positive semi-definite case.

PART I

LEAST-INDEX RESOLUTION OF DEGENERACY IN THE DANTZIG-COTTLE PRINCIPAL PIVOTING METHOD

1. Introduction

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the <u>linear complementarity problem</u> (q,M) is that of finding a solution to the system

(1) $$w = q + Mz$$

(2) $$w \geq 0, \quad z \geq 0$$

(3) $$w^T z = 0$$

(or showing that no solution exists). A pair (w,z) of n-vectors is a <u>complementary solution</u> of (1) provided

$$z_1 w_1 = 0,\ z_2 w_2 = 0,\ \ldots,\ z_n w_n = 0 .$$

A <u>basic set</u> of variables consists of any ordered set of n variables w_i and z_j such that their coefficient matrix in (1), called a <u>basis</u>, is nonsingular. A <u>complementary basic set</u> of variables is one in which exactly one variable of each complementary pair (w_i, z_i) is basic. Finally, a <u>basic solution</u> is the one found by solving for the value of a given set of basic variables when the nonbasic variables are set equal to zero.

A basic solution of the equation (1) is said to be <u>degenerate</u> if at least one of the basic variables equals zero. As in simplicial methods for linear and quadratic programming, degeneracy also causes difficulties in

simplicial methods for the linear complementarity problem (LCP). More precisely, degeneracy can lead to the phenomenon known as circling: a sequence of bases which (after finitely many steps) repeats itself. Degeneracy per se is not the problem; but when it is present, circling is a possibility and must be avoided if the simplicial methods are to work and be finite. The "degeneracy problem" refers to the difficulties associated with circling.

Except for Murty's scheme [15], the only theoretical techniques available for handling the degeneracy problem in the LCP have been lexicographic (perturbation) pivot selection rules and random choice rules. We shall not review them here; they can be found in the work of Zoutendijk [20, pp. 80-90], Graves [10], Eaves [8] and Lemke [12].

In this part, we present a natural least-index pivot selection rule which, when imposed on the Dantzig-Cottle principal pivoting method [2], [5], [7] for the LCP, will ensure its finiteness. This rule is also very easy to implement and does not require extra storage or computation.

2. The PPM with P-matrices

The Dantzig-Cottle principal pivoting method (PPM) was devised to process (q,M) where M is either a P-matrix or a positive semi-definite matrix. The matrix M is called a P-matrix if the determinants of all its principal submatrices are positive. We consider P-matrices in this section and leave the case of positive semi-definite matrices to Section 3.

In applying the least-index pivot selection rule to resolve the degeneracy problem in the PPM, we rely on a paper of Murty [15] which also considers a least-index rule in the framework of a Bard-type algorithm for

the LCP with P-matrices. For the references on the Bard-type algorithm, see Bard [1, pp. 146-151], Stickney and Watson [18], and Zoutendijk [20, pp. 80-90].

2.1. Murty's scheme for (q,M) where M is a P-matrix

It is well-known that when M is a P-matrix, (q,M) has a unique solution for every q [16], [17]. Using this fact, Murty [15] proposed the following finite scheme:

Murty' scheme.

Step 0. Set $h = 0$. Begin with the system $w^h = q^h + M^h z^h$ where $w^0 = q^0 + M^0 z^0$ is the given system $w = q + Mz$.

Step 1. If $q^h \geq 0$, stop. $[w^h;z^h] = [q^h;0]$ is the solution. Otherwise

Step 2. Choose $k = \min\{i \mid q_i^h < 0\}$ and pivot on m_{kk}^h, i.e., z_k^h is brought into the basic set in place of w_k^h. Set $h = h+1$ and return to Step 1.

Murty proves that this scheme solves (q,M). We emphasize that the finiteness of this scheme is a result of the uniqueness of the solution to (q,M) where M is a P-matrix. It is also interesting to note that his proof can be applied to show the following:

Proposition. In Murty's scheme, a pivot in row k must be followed by a pivot in some row with a larger index before another pivot in row k can occur.

2.2. A brief review of the principal pivoting method

There are two versions of the PPM: the symmetric and the asymmetric versions. They both make use of the invariance of P-matrices under principal pivoting [7], [19].

Consider an LCP (q,M) in which M is a P-matrix. In the following, w^h, z^h, M^h and q^h represent the basic vector, the non-basic vector, the matrix and the constant column at the h-th interation, respectively.

Symmetric version of the PPM [2], [4], [5], [7].

Step 0. Set $h = 0$. Begin with the system $w^h = q^h + M^h z^h$ where $w^0 = q^0 + M^0 z^0$ denotes the given system $w = q + Mz$.

Step 1. If $q^h \geq 0$, stop. $[w^h;z^h] = [q^h;0]$ is the solution. Otherwise choose some $q_s^h < 0$. Call w_s^h the distinguished variable and z_s^h the driving variable.

Step 2. Determine the blocking variable by letting θ be the largest value of the driving variable z_s^h such that

$$w_s^h = q_s^h + m_{ss}^h z_s^h \leq 0$$

$$w_i^h = q_i^h + m_{is}^h z_s^h \geq 0 , \qquad \text{if } q_i^h \geq 0 > m_{is}^h .$$

Step 3. If $\theta = -q_s^h/m_{ss}^h$, i.e., z_s^h is blocked by w_s^h, then pivot on m_{ss}^h. Replace h by $h+1$ and return to Step 1. If $-q_s^h/m_{ss}^h > \theta = -q_t^h/m_{ts}^h$ for some t where $q_t^h \geq 0 > m_{ts}^h$, i.e. z_s^h is blocked by w_t^h, then pivot on m_{tt}^h. Replace h by $h+1$ and return to Step 2.

In this algorithm, each return to Step 1 marks the completion of a major cycle. Assuming nondegeneracy, Dantzig and Cottle [7] showed that during a major cycle, the distinguished variable increases to zero in a finite number of steps. However, the completion of a major cycle reduces the number of negative basic variables by at least one. Therefore, no more than n major cycles are required to obtain a solution of (q,M).

The asymmetric version is the same as the symmetric version except that in Step 3, if the distinguished variable is not blocking and, say, w_t^h is blocking, then one performs the pivot on m_{ts}^h and returns to Step 2 with $z_s^{h+1} = z_t^h$, the complement of the exiting basic variable, as the new driving variable.

NOTE: In the following, we shall use the notation $\langle w_i, z_j \rangle$ to represent that a pivot on m_{ij} is performed thereby making z_j basic in place of w_i.

The two versions of the PPM are closely related. In fact, we have the following:

Theorem 1. Under the same rule to break ties among the blocking variables, the symmetric and asymmetric versions of a major cycle generate the same sequence of exiting basic variables.

Proof. Without loss of generality, we may assume that z_1 is the driving variable at the start of a major cycle. If w_1 blocks z_1, there is nothing to prove. Hence suppose w_k blocks z_1 for some $k \neq 1$. Under the same rule to break ties, the two versions have the same exiting variable at this step. In particular, the corresponding variables in the two versions have the same values at this step.

In the symmetric version, we perform the pivot $\langle w_k, z_k \rangle$ after which the system can be written as

$$(4) \qquad z_k = -\frac{q_k}{m_{kk}} + \frac{1}{m_{kk}} w_k - \frac{1}{m_{kk}} \sum_{\substack{j=1 \\ j \neq k}}^{n} m_{kj} z_j$$

$$(5) \qquad w_i = \left(q_i - \frac{m_{ik}}{m_{kk}} q_k\right) + \frac{m_{ik}}{m_{kk}} w_k + \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(m_{ij} - \frac{m_{ik} m_{kj}}{m_{kk}}\right) z_j$$

for all $i \neq k$.

In the asymmetric version, we perform $\langle w_k, z_1 \rangle$. After the pivot z_k becomes the new driving variable and the system becomes

$$(6) \qquad z_1 = -\frac{q_k}{m_{k1}} + \frac{1}{m_{k1}} w_k - \frac{1}{m_{k1}} \sum_{j=2}^{n} m_{kj} z_j$$

$$(7) \qquad w_i = \left(q_i - \frac{m_{i1}}{m_{k1}} q_k\right) + \frac{m_{i1}}{m_{k1}} w_k + \sum_{\substack{j=2 \\ j \neq k}}^{n} \left(m_{ij} - m_{i1} \frac{m_{kj}}{m_{k1}}\right) z_j + \left(m_{ik} - m_{i1} \frac{m_{kk}}{m_{k1}}\right) z_k$$

for all $i \neq k$.

Now, since z_1 and z_k are the driving variables in the two versions, respectively, we compare $m_{i1} - (m_{ik} m_{k1})/m_{kk}$, the coefficient

of z_1 in (5), and $m_{ik} - (m_{i1}m_{kk})/m_{k1}$, the coefficient of z_k in (7). We can rewrite

$$m_{ik} - \frac{m_{i1}m_{kk}}{m_{k1}} = \frac{m_{kk}}{-m_{k1}}\left(m_{i1} - \frac{m_{ik}m_{k1}}{m_{kk}}\right) \qquad \text{for all } i \neq k .$$

Since w_k blocked z_1 at the previous step, we have $m_{k1} < 0$. Also $m_{kk} > 0$ since M is a P-matrix. Therefore $m_{ik} - (m_{i1}m_{kk})/m_{k1}$ and $m_{i1} - (m_{ik}m_{k1})/m_{kk}$ are of the same sign for all $i \neq k$. Clearly $-m_{k1}/m_{kk}$, the coefficient of z_1 in (4), and $-m_{kk}/m_{k1}$, the coefficient of z_k in (6), are both positive. Thus the two driving columns at this step have the same sign configuration. Also note that before the pivots $\langle w_k, z_k\rangle$ and $\langle w_k, z_1\rangle$, the corresponding variables in the two versions have the same values.

Now, suppose w_t is exiting in the symmetric version at the next step and suppose blocking occurs when $z_1 = \bar{z}_1$. Then by (5),

$$(8) \qquad 0 = w_t = \left(q_t - \frac{m_{tk}}{m_{kk}} q_k\right) + \left(m_{t1} - \frac{m_{tk}m_{k1}}{m_{kk}}\right)\bar{z}_1 \quad .$$

Also by (4),

$$z_k = -\frac{q_k}{m_{kk}} - \frac{m_{k1}}{m_{kk}}\bar{z}_1 \ .$$

Let us consider the system in the asymmetric version when

$$z_k = \bar{z}_k \equiv -\frac{q_k}{m_{kk}} - \frac{m_{k1}}{m_{kk}} \bar{z}_1 ,$$

and

$$z_2 = z_3 = \cdots = z_{k-1} = w_k = z_{k+1} = \quad = z_n = 0 .$$

By (7) and (8),

$$\begin{aligned} w_t &= \left(q_t - m_{t1} \frac{q_k}{m_{k1}}\right) + \left(m_{tk} - m_{t1} \frac{m_{kk}}{m_{k1}}\right) \bar{z}_k \\ &= q_t - m_{t1} \frac{q_k}{m_{k1}} + \left(m_{tk} - m_{t1} \frac{m_{kk}}{m_{k1}}\right)\left(\frac{-m_{k1}}{m_{kk}}\right)\left(\bar{z}_1 + \frac{q_k}{m_{k1}}\right) \\ &= \left(q_t - \frac{m_{tk}}{m_{kk}} q_k\right) + \left(m_{t1} - \frac{m_{tk} m_{k1}}{m_{kk}}\right) \bar{z}_1 \\ &= 0 \end{aligned}$$

and clearly z_1 has the value $\bar{z}_1$. It follows that, at this step, w_t is also a blocking variable in the asymmetric version. Furthermore, the basic variables in the two versions are the same except that z_k is basic in the symmetric version while z_1 is basic in the asymmetric version. Therefore, under the same rule to break ties, w_t is also the exiting variable in the asymmetric version.

The pivoting sequence $\langle w_k, z_1\rangle$, $\langle w_t, z_k\rangle$ of the asymmetric version has the same effect as the pivoting sequence $\langle w_k, z_k\rangle$, $\langle w_t, z_1\rangle$. In the symmetric version of the corresponding pivoting sequence is $\langle w_k, z_k\rangle$, $\langle w_t, z_t\rangle$. Thus, the second pivot in each version can be regarded as the first pivot associated with the corresponding method applied to the principal

transform of (q,M) obtained by the principal pivot $\langle w_k, z_k \rangle$. Therefore the argument just given applies and the two versions have the same exiting basic variable at the following step.

Generally, suppose a sequence of pivots has been performed in the asymmetric version and the last pivot is on m_{ij}, this pivoting sequence has the same effect as a block pivot on some principal submatrix M_{SS} followed by a pivot on m_{ii}. In the symmetric version, the corresponding pivoting sequence can be regarded as obtained by performing a block pivot on the principal submatrix M_{SS} followed by a pivot on m_{ii}. Thus, the last pivot in each version can be regarded as the first pivot associated with the corresponding method applied to the principal transform of (q,M) obtained by a block pivot on the principal submatrix M_{SS}. Therefore by the same argument as before, the two versions have the same exiting basic variable at the next step. □

Remark. It follows from Theorem 1 that if one version is finite, then so is the other. Accordingly, we work with the symmetric version only.

2.3. The PPM with the least-index rule

If degeneracy occurs, the PPM may lead to circling. Some circling examples will be given in Section 4. We consider here a least-index rule which when imposed in the PPM, will ensure its finiteness.

Least-index rule.

In applying the PPM to solve (q,M) break ties among the blocking variables as follows:

(A) If the distinguished variable is blocking, choose it as the exiting variable (and the major cycle terminates).

(B) Otherwise, choose the blocking variable with the smallest index as the exiting variable.

In the following, we show that the PPM with this least-index rule will solve the problem in a finite number of steps. To do so, it suffices to show that each major cycle is finite. We shall prove this by first assuming that circling occurs in a major cycle and then deriving a contradiction. Without loss of generality, we may assume that w_1 is the distinguished variable in this major cycle. Let

$$H = \{h_1, h_2, \ldots, h_m\}$$

where $h_{j+1} = h_j+1$ for $j = 1,2,\ldots,m-1$, and $w^{h_1} = q^{h_1} + M^{h_1}z^{h_1}$ represents the system in which a previous basic set is repeated for the first time. h_m is a positive integer such that the system $w^{h_m+1} = q^{h_m+1} + M^{h_m+1}z^{h_m+1}$ has the same basic set as that of $w^{h_1} = q^{h_1} + M^{h_1}z^{h_1}$. Note that $w_1^h = w_1$ and $z_1^h = z_1$ for all $h = h_1,\ldots,h_m$. In the following, the phrase "during circling" will mean "during the pivoting steps $h_1, h_2, \ldots, h_m$."

Lemma 1. If circling occurs, then the value of the driving variable z_1^h is fixed during circling.

Proof. Since z_1^h is the driving variable, it is nondecreasing in this major cycle. If z_1^h is not fixed during circling, then it attains two values $\bar{z}_1 < \bar{\bar{z}}_1$. There are only finitely many basic solutions, each of which corresponds to a unique set of values of the basic variables. Now since z_1^h increases from $\bar{z}_1$ to $\bar{\bar{z}}_1$, we can not return to a previously encountered basis, a contradiction. □

Lemma 2. Assume circling occurs. Let $K = \{i | w_i^h$ becomes nonbasic during circling$\}$. Then during circling the values of the basic variables w_i^h are at their lower bounds (zero) for all $i \in K$.

Proof. The set K can be written as $K = \{k_1, k_2, \ldots, k_m\}$ where k_j is the index of the variable leaving the basic set at step h_j. Since $w_{k_1}^{h_1}$ is a blocking variable at the first step of circling, it follows from Lemma 1 that $w_{k_1}^{h_1} = 0$. Therefore after the pivot $\langle w_{k_1}^{h_1}, z_{k_1}^{h_1} \rangle$, all other basic variables remain fixed.

Now the algorithm tries to increase $z_1^{h_2}$ again. (Note that $z_1^{h_1} = z_1^{h_2} = z_1$.) However, Lemma 1 implies that it can not be increased. Thus $w_{k_2}^{h_2} = 0$. Since each w_k^h is involved in a pivot during circling, the argument just given applies and the proof is complete. □

Lemma 3. If circling occurs, then at each step of circling $m_{k1}^h < 0$ for some $k \in K$.

Proof. By the definition of the set K, at each setp of circling, w_k^h becomes nonbasic for some $k \in K$. This implies that w_k^h is a blocking variable at this step. However, since z_1^h is the driving variable, this can happen only when $m_{k1}^h < 0$. □

Now we come to our result.

Theorem 2. In the PPM with the least-index rule applied to (q,M) where M is a P-matrix, every major cycle consists of a finite number of pivots.

Proof. Suppose circling occurs in a major cycle in which z_1^h is the driving variable. Lemmas 2 and 3 imply that, during circling, the algorithm looks for the index j where

$$j = \min\{i \mid w_i^h = 0 \text{ and } m_i^h < 0\}$$

and then performs $\langle w_j^h, z_j^h\rangle$. Therefore, during circling, the PPM with the least index rule is merely Murty's scheme on the LCP $(M_{K1}^{h_1}, M_{KK}^{h_1})$ where K is as defined in Lemma 2. However, Murty's scheme is finite. This implies that after a finite number of steps $M_{K1}^h \geq 0$ for some h, in contradiction to Lemma 3. □

Corollary. The PPM with the least-index rule applied to the LCP (q,M), where M is a P-matrix, will find the solution in a finite number of steps.

Proof. The completion of each major cycle reduces the number of negative basic variables by at least one, and by Theorem 2 each major cycle is finite. □

__Remark.__ If an algorithm just changes the basis and leaves the values of all variables fixed during some consecutive steps, we say that __stalling__ occurs in these steps. The proof of Theorem 2 and the Proposition in Section 2.1 show that during stalling a pivot in row k must be followed by a pivot in some row with a larger index before another pivot in row k can occur.

3. The PPM with positive semi-definite matrices

3.1. Statement of the method

When the system $w = q + Mz$, $w \geq 0$, $z \geq 0$, has a solution, we say that (q,M) is __feasible__, otherwise it is __infeasible__. It is well-known that when M is positive semi-definite, (q,M) has a solution whenever it is feasible. With some modifications, the PPM as stated in Section 2.2 can be applied to find a solution of (q,M) or to detect its infeasibility. Similar to the case of P-matrices, all variables whose current value is non-negative are bounded below by zero. Moreover, those variables whose current value is negative will be bounded below by a fixed negative number β (the same one for all such variables). For example, β can be chosen as any negative number such that $\beta < \min_i\{q_i\}$ if $q \not\geq 0$. Accordingly, we modify our notion of a __basic solution__ to allow nonbasic variables to assume the value 0 or β. The value β arises from the situation where a basic variable decreases to β, thereby blocking the driving variable. The method will make that blocking variable nonbasic at value β. This device is necessary. For example, if

$$q = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

and w_1 is distinguished, then z_1 is unblocked. Yet the problem has a solution. We shall also change the definition of a nondegenerate solution to be one in which at most n of the $2n$ variables equal 0 or β.

Notation.

As before, the superscript h denotes the iteration number. Since the nonbasic variables may attain the negative value β, we use $w_i^h(\bar{z}_1^h, \ldots, \bar{z}_n^h)$ to denote the value of the basic variable w_i^h when the nonbasic variables $z_1^h, \ldots, z_n^h$ have the values $\bar{z}_1^h, \ldots, \bar{z}_n^h$.

Symmetric version of the method [2], [4], [5], [7]

(Note: Nondegeneracy is assumed here.)

Step 0. Set $h = 0$. Begin with the system $w^h = q^h + M^h z^h$ and the solution $[\bar{w}^h; \bar{z}^h] = [q^h; 0]$, where $w^0 = q^0 + M^0 z^0$ denotes the given system $w = q + Mz$.

Step 1. If $q^h \geq 0$, stop. $[\bar{w}^h; \bar{z}^h] = [q^h;0]$ is a solution. If neither q^h nor $[\bar{w}^h; \bar{z}^h]$ is nonnegative, choose some $\bar{w}_s^h < 0$ or $\bar{z}_s^h = \beta$. Call w_s^h the distinguished variable and z_s^h the driving variable.

Step 2. Let θ^h be the largest value of $z_s^h \geq \bar{z}_s^h$ satisfying the following conditions:

(i) $z_s^h \leq 0$ if $\bar{z}_s^h = \beta$

(ii) $w_s^h(\bar{z}_1^h, \ldots, \bar{z}_{s-1}^h, z_s^h, \bar{z}_{s+1}^h, \ldots, \bar{z}_n^h) \leq 0$ if $\bar{w}_s^h < 0$

(iii) $w_i^h(\bar{z}_1^h, \ldots, \bar{z}_{s-1}^h, z_s^h, \bar{z}_{s+1}^h, \ldots, \bar{z}_n^h) \geq 0$ if $\bar{w}_i^h \geq 0$

(iv) $w_i^h(\bar{z}_1^h, \ldots, \bar{z}_{s-1}^h, z_s^h, \bar{z}_{s+1}^h, \ldots, \bar{z}_n^h) \geq \beta$ if $\bar{w}_i^h < 0$.

Step 3. If $\theta^h = \infty$, i.e. the driving variable z_s^h is unblocked, stop. No feasible solution exists. If $\theta^h = 0$, i.e. the driving variable z_s^h blocks itself, then put $\bar{z}_s^{h+1} = 0$, $\bar{z}_i^{h+1} = \bar{z}_i^h$ for $i \neq s$ and $\bar{w}^{h+1} = w^h(\bar{z}_1^h, \ldots, \bar{z}_{s-1}^h, 0, \bar{z}_{s+1}^h, \ldots, \bar{z}_n^h)$. Return to Step 1 with h replaced by $h+1$. If $0 < \theta^h < \infty$, let t be the unique index determined by the conditions (ii), (iii) and (iv) of Step 2.

Step 4. If $m_{tt}^h > 0$ and $t = s$, pivot $\langle w_s^h, z_s^h \rangle$ and return to Step 1 with h replaced by $h+1$.
If $m_{tt}^h > 0$ and $t \neq s$, pivot $\langle w_t^h, z_t^h \rangle$ and return to Step 2 with h replaced by $h+1$.
If $m_{tt}^h = 0$, perform a block pivot of order 2 on the principal submatrix M_{SS}^h, where $S = \{s,t\}$, and return to Step 2 with h replaced by $h+1$.

In this algorithm, each return to Step 1 marks the completion of a <u>major cycle</u>. Under the assumption of nondegeneracy, the driving variable and the distinguished variable are always increasing while their

sum is strictly increasing [4]. Thus after finitely many pivots within a major cycle, the negative distinguished variable increases to zero, or else it is detected that the problem has no feasible solution. Furthermore, the end of a major cycle reduces the number of negative components in (w,z) by at least one. Therefore the method is finite.

There is an asymmetric version of the above method. It uses simple pivots at each step to exchange the blocking variable with the driving variable and takes the complement of the blocking variable as the new driving variable. By a proof similar to that of Theorem 1, it can be shown that, under the same tie-breaking rules, the two versions have the same sequence of exiting basic variables in a major cycle. (Except that when termination of this major cycle occurs, the initial driving variable may be the exiting variable in the symmetric version while its complement is exiting in the symmetric version.) Accordingly, we work with the symmetric version only.

3.2. The least-index rule

When degeneracy occurs, the above method may circle. In this section we show that with the least-index rule of Section 2, the symmetric version of the PPM will process (q,M), where M is positive semi-definite, in a finite number of steps. In other words, it will either find a solution or detect the infeasibility of the problem. Again, it suffices to show that each major cycle is finite.

Suppose circling occurs in a major cycle in which w_1 is the distinguished variable. Then, as in the case when M is a P-matrix, since w_1 and z_1 are monotonically increasing, both w_1 and z_1 are

fixed during circling. However, the algorithm tries to increase z_1 or w_1 in this major cycle. Therefore, as in the case when M is a P-matrix, stalling occurs in these steps. Accordingly, if we delete all the variables that are not involved in the pivoting during circling, the PPM with the least-index rule merely looks for the index i, where

$$i = \min\{j \mid m^h_{j1} < 0\}$$

and then the PPM pivots on m^h_{ii} (if $m^h_{ii} \neq 0$) or it pivots on

$$\begin{bmatrix} m^h_{11} & m^h_{1i} \\ m^h_{i1} & m^h_{ii} \end{bmatrix} \qquad (\text{if } m^h_{ii} = 0) .$$

Without loss of generality we may assume that all the variables are involved in the pivoting during circling. Then, during circling, the PPM with the least-index rule performs the same pivoting sequence as the following scheme does.

Scheme

Step 0. Start with the system $w^h = q^h + M^h z^h$, $h = 0$, where $w^0 = q^0 + M^0 z^0$ is the initial system. (In the following, $M^h_{\cdot i}$ represents the column of M^h corresponding to the nonbasic variable z^h_i at the iteration h. Similarly, $M^h_{i\cdot}$ represents the row of M^h corresponding to the basic variable w^h_i.)

Step 1. If $M_{.1}^h \geq 0$, stop. The driving variable z_1^h can be increased strictly. Otherwise, let $k = \min\{i | m_{i1}^h < 0\}$.

Step. 2. If $m_{kk}^h > 0$, perform a pivot on m_{kk}^h and return to Step 1 with h replaced by $h+1$. Otherwise, perform a block pivot of order 2 on the principal submatrix

$$\begin{bmatrix} m_{11}^h & m_{1k}^h \\ m_{k1}^h & m_{kk}^h \end{bmatrix}$$

and return to Step 1 with h replaced by $h+1$.

If we can show that $M_{.1}^h \geq 0$ after a finite number of pivots in the above scheme, then, since the driving variable z_1^h can be increased strictly at this step, we obtain a contradiction to the assumption that circling occurs in a major cycle (in which w_1 is the distinguished variable) of the PPM with the least-index rule.

Lemma 4. In the above scheme, a pivot in row k, where $2 \leq k \leq n$, must be followed by a pivot in some row with a larger index before another pivot in row k can occur.

Proof. We will prove this by induction. If the matrix M is of order 1 or 2, the lemma is trivial. Suppose the lemma holds when the order of M is less than n and now consider the case when M is of order n.

Let us examine the situation where two pivots occur in row k and $2 \leq k \leq n-1$. If between these two pivots, there is no pivot in some

row with a larger index, then by deleting $M_{\cdot n}$ and $M_{n\cdot}$, a contradiction to the inductive hypothesis will be derived. Therefore, it suffices to show that there is at most one pivot in row n.

Suppose a pivot occurs in row n at iteration h_1. Let (T1) be the corresponding schema at this iteration.

$$\text{(T1)} \qquad \begin{array}{c} \\ w_1 = \\ \vdots \\ w_n = \end{array} \begin{array}{c|ccc|} \multicolumn{1}{c}{1} & z_1 & \cdots & \multicolumn{1}{c}{z_n} \\ \hline q_1 & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & & \vdots \\ q_n & m_{n1} & \cdots & m_{nn} \\ \hline \end{array}$$

By the choice of the pivot row, $m_{i1} \geq 0$ for all $i \leq n-1$ and $m_{n1} < 0$ in (T1). Note that $w_1, \ldots, w_n$ are the basic variables in (T1).

Suppose the next occurrence of a pivot in row n is at iteration h_2. When this occurs, z_n must be the exiting basic variable and w_1 is either basic (Case I) or nonbasic (Case II).

<u>Case I</u>. w_1 is a basic variable at iteration h_2.

Let S be the set of indices i such that w_i is nonbasic at iteration h_2. Note that $1 \notin S$. Let $\bar{M}$ denote the principal transform of M at this iteration. Clearly $\bar{M}$ can be obtained from M by performing a block pivot on the principal submatrix M_{SS}. Thus $\bar{M}_{S1} = -M_{SS}^{-1}M_{S1}$ (since $1 \notin S$) and therefore

$$M_{S1}^T\bar{M}_{S1} = -M_{S1}^T M_{SS}^{-1} M_{S1} \leq 0$$

since M_{SS}^{-1} is positive semi-definite. However, since $m_{i1} \geq 0$, $\bar{m}_{i1} \geq 0$ for all $i < n$ and $m_{n1} < 0$, $\bar{m}_{n1} < 0$, we have $M_{S1}^T \bar{M}_{S1} > 0$ (since $n \in S$), a contradiction.

<u>Case II</u>. w_1 is a nonbasic variable at iteration h_2.

We shall use the same notation as Case I. Note that $1 \in S$ in this case. Since $\bar{M}$ is positive semi-definite, $\bar{m}_{11} > 0$ or $\bar{m}_{11} = 0$.

<u>Case II.1</u>. $\bar{m}_{11} > 0$.

By performing a pivot on $\bar{m}_{11}$, w_1 becomes a basic variable and the sign configuration of $\bar{M}_{\cdot 1}$ is unchanged. In other words, $\bar{m}_{i1} \geq 0$ for all $i \leq n-1$ and $\bar{m}_{n1} < 0$. Since w_1 is a basic variable now, as in Case I, a contradiction can be derived.

<u>Case II.2</u>. $\bar{m}_{11} = 0$.

<u>Case II.2.1</u>. $m_{11} > 0$.

By performing a pivot on m_{11} in schema (T1), w_1 becomes a non-basic variable and the sign configuration of $M_{\cdot 1}$ is unchanged. Therefore, as in Case I, a contradiction can be derived.

Case II.2.2. $m_{11} = 0$.

Let us denote the schema at Step h_2 as (T2)

$$
\text{(T2)} \qquad
\begin{array}{r|c|c:c|}
 & 1 & w_S & z_{\tilde{S}} \\
\hline
z_S = & \bar{q}_S & \bar{M}_{SS} & \bar{M}_{S\tilde{S}} \\
\hdashline
w_{\tilde{S}} = & \bar{q}_{\tilde{S}} & \bar{M}_{\tilde{S}S} & \bar{M}_{\tilde{S}\tilde{S}} \\
\hline
\end{array}
$$

where $\tilde{S} = \{1,\ldots,n\} \setminus S$. Note that (T2) can be considered as obtained from (T1) by performing a block pivot on the principal submatrix M_{SS}. (Recall that $1 \in S$ in this case.)

If we enlarge the schema (T1) to (T1*) by adding one row and one column such that $M_{n+1.} = (1, 0, \ldots, 0, 1)$ and $M_{.n+1} = (-1, 0, \ldots, 0, 1)^T$ and q_{n+1} an arbitrary number, the enlarged matrix of order n+1 is still positive semidefinite.

$$
\text{(T1*)} \qquad
\begin{array}{r|c:cccc|c|}
 & 1 & z_1 & & \cdots & z_n & z_{n+1} \\
\hline
w_1 = & q_1 & m_{11} & & \cdots & m_{1n} & -1 \\
 & & & & & & 0 \\
\vdots & \vdots & \vdots & & & \vdots & \vdots \\
w_n = & q_n & m_{n1} & & \cdots & m_{nn} & 0 \\
\hline
w_{n+1} = & q_{n+1} & 1 & 0 & \cdots & 0 & 1 \\
\hline
\end{array}
$$

By performing a block pivot on the principal submatrix M_{SS} in (T1*), a schema (T2*) is obtained which has the same entries $\bar{q}_i$, $\bar{m}_{ij}$ as the schema (T2) for all $i, j \le n$.

$$
(\mathrm{T2}^*)\qquad
\begin{array}{rc|c|cc|c|}
 & & 1 & w_S & z_{\tilde{S}} & z_{n+1} \\
\hline
z_S & = & \bar{q}_S & \bar{M}_{SS} & \bar{M}_{S\tilde{S}} & \bar{M}_{S,n+1} \\
w_{\tilde{S}} & = & \bar{q}_{\tilde{S}} & \bar{M}_{\tilde{S}S} & \bar{M}_{\tilde{S}\tilde{S}} & \bar{M}_{\tilde{S},n+1} \\
\hline
w_{n+1} & = & \bar{q}_{n+1} & M_{n+1,S} & \bar{M}_{n+1,\tilde{S}} & \bar{M}_{n+1,n+1} \\
\hline
\end{array}
$$

Also, (T2*) has the same basic variables as (T2) does. Therefore (T2*) is also an enlargement of the schema (T2). By pivotal algebra

$$
\begin{aligned}
\bar{m}_{n+1,1} &= M_{n+1,S} \cdot \bar{M}_{S1} \\
&= (1,0,\ldots,0) \cdot \bar{M}_{S1} \\
&= \bar{m}_{11} \\
&= 0 .
\end{aligned}
$$

Furthermore, by performing a block pivot on the principal submatrix

$$
\begin{bmatrix} m_{11} & m_{1,n+1} \\ m_{n+1,1} & m_{n+1,n+1} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}
$$

the schema (T1*) becomes the following schema (T2**) in which w_1 and w_{n+1} are nonbasic while all other w_i's are still basic.

$$
(\mathrm{T2}^{**})\qquad
\begin{array}{cc|c|ccccc|}
 & & 1 & w_1 & z_2 & & z_n & w_{n+1} \\
\hline
z_1 & = & \bar{\bar{q}}_1 & \bar{\bar{m}}_{11} & \bar{\bar{m}}_{12} & \cdots & \bar{\bar{m}}_{1n} & \bar{\bar{m}}_{1,n+1} \\
w_2 & = & \bar{\bar{q}}_2 & \bar{\bar{m}}_{21} & \bar{\bar{m}}_{22} & \cdots & \bar{\bar{m}}_{2n} & \bar{\bar{m}}_{2,n+1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
w_n & = & \bar{\bar{q}}_n & \bar{\bar{m}}_{n1} & \bar{\bar{m}}_{n2} & \cdots & \bar{\bar{m}}_{nn} & \bar{\bar{m}}_{n,n+1} \\
z_{n+1} & = & \bar{\bar{q}}_{n+1} & \bar{\bar{m}}_{n+1,1} & \bar{\bar{m}}_{n+1,2} & \cdots & \bar{\bar{m}}_{n+1,n} & \bar{\bar{m}}_{n+1,n+1} \\
\hline
\end{array}
$$

Since $m_{11} = 0$, $m_{i1} \geq 0$ for all $2 \leq i < n$, $m_{n1} < 0$ and $m_{n+1,1} = 1$, it can easily be seen that $\bar{\bar{m}}_{11} = 1$, $\bar{\bar{m}}_{i1} \geq 0$ for all $2 \leq i < n$, $\bar{\bar{m}}_{n1} < 0$ and $\bar{\bar{m}}_{n+1,1} = -1$

Now since both (T2*) and (T2**) are principal transforms of the schema (T1*), (T2**) is a principal transform of (T2*). In fact, if we denote $R = (S \setminus \{1\}) \cup \{n+1\}$, then (T2**) can be obtained by performing a block pivot on the principal submatrix $\bar{M}_{RR}$ in (T2*). Therefore $\bar{\bar{M}}_{R1} = -\bar{M}_{RR}^{-1}\bar{M}_{R1}$, and thus $\bar{M}_{R1}^T\bar{\bar{M}}_{R1} = -\bar{M}_{R1}^T\bar{M}_{RR}^{-1}\bar{M}_{R1}$. Since $\bar{M}_{RR}^{-1}$ is positive semi-definite, $-\bar{M}_{R1}^T\bar{M}_{RR}^{-1}\bar{M}_{R1} \leq 0$. However, since $n \in R$, $n+1 \in R$ and $\bar{m}_{n1} < 0$, $\bar{m}_{n+1,1} = 0$, $\bar{\bar{m}}_{n1} < 0$, $\bar{\bar{m}}_{n+1,1} = -1$ while for other $i \in R$, $\bar{m}_{i1} \geq 0$, $\bar{\bar{m}}_{i1} \geq 0$; therefore $\bar{M}_{R1}^T\bar{\bar{M}}_{R1} > 0$, a contradiction. □

<u>Lemma 5</u>. In the above schema, $M_{\cdot 1} \geq 0$ after a finite number of iterations.

<u>Proof</u>. For $j > 1$, let $\nu(j)$ be the number of pivots that occur in row j. In the proof of Lemma 4, we have shown that $\nu(n) \leq 1$.

Furthermore, it follows from Lemma 4 that

$$\nu(j) \leq \sum_{i=j+1}^{n} \nu(i) + 1 .$$

In other words,

$$\nu(n-1) \leq \nu(n) + 1 \leq 2$$

$$\nu(n-2) \leq 2^2$$

$$\vdots$$

$$\nu(n-i) \leq 2^{i-1} + 2^{i-2} + \cdots + 2 + 2^0 + 1 = 2^i .$$

Therefore, the above scheme will terminate after a finite number of iterations. □

Theorem 3. In the positive semi-definite case, every major cycle of the PPM with least-index rule consists of a finite number of pivots.

Proof. Suppose circling occurs in a major cycle in which w_1 is the distinguished variable. Then, as in the case when M is a P-matrix, since w_1 and z_1 are monotonically increasing, both w_1 and z_1 are fixed during circling. However, it follows from Lemma 5 that $M_{\cdot 1} \geq 0$ after a finite number of steps. Therefore either w_1 or z_1 can be strictly increased after a finite number of steps, in contradiction to the assumption that circling occurs. □

Corollary. In the positive semi-definite case, the PPM with least-index rule will process the problem in a finite number of steps.

Proof. Since each major cycle reduces the number of negative components in (w,z) by at least one, the result follows from Theorem 3. □

Remark 1. As in the case when M is a P-matrix, Lemma 4 implies that if stalling occurs in the PPM with least-index rule for (q,M) where M is positive semi-definite, then during stalling, a pivot in row k (except the row corresponding to the distinguished variable) must be followed by a pivot in some row with a larger index before another pivot in row k can occur.

Remark 2. The least-index rule states that if the distinguished variable is blocking, then it is chosen as the exiting basic variable even if there is a blocking basic variable with a smaller index. This is essential in the interpretation of the rule. The following is an example which has a solution, but if the least-index rule is incorrectly applied, the driving variable will be unblocked and hence give the false impression that the problem is infeasible.

Example. Consider (q,M) where $q = (1, -1, -1)^T$ and

$$M = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly, M is positive-semi-definite and $(w;z) = (0, 0, 0; 0, 1, 1)$ is a solution. Consider the major cycle in which w_2 is the distinguished variable.

$$
\begin{array}{cc|c|ccc|}
 & & 1 & z_1 & z_2 \uparrow & z_3 \\
\hline
w_1 & = & 1 & 1 & -1 & 0 \\
w_2 & = & -1 & -1 & 1 & 0 \\
w_3 & = & -1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

When the driving variable z_2 increases to 1, both w_1 and w_2 are blocking. If our least-index rule is imposed, then the pivot $\langle w_2, z_2 \rangle$ is performed since w_2 is the distinguished variable. However, if the least-index rule is incorrectly applied, w_1 is chosen as the exiting basic variable, and then the pivot $\langle w_1, z_1 \rangle$ is performed.

$$
\begin{array}{cc|c|ccc|}
 & & 1 & w_1 & z_2 \uparrow & z_3 \\
\hline
z_1 & = & -1 & 1 & 1 & 0 \\
w_2 & = & 0 & -1 & 0 & 0 \\
w_3 & = & -1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

Now, since w_2 is still the distinguished variable, the driving variable is still z_2 and its column is nonnegative, hence z_2 is unblocked. Therefore, according to Step 3 of the statement of the PPM in Section 3.1, one gets the mistaken impression that the problem is infeasible.

4. <u>Circling examples of minimal dimension.</u>

In this section, we give a circling example for the PPM on (q,M) where M is a positive definite matrix of order four (hence M is a P-matrix as well as a positive semi-definite matrix). We will also show that four is the sharp lower bound on the order of M for the circling to occur.

Example 1. Consider the initial schema in which w_1 is the distinguished variable

$$\begin{array}{cc|c|cccc|}
 & & 1 & z_1 & z_2 & z_3 & z_4 \\
\cline{3-7}
w_1 & = & -1 & 1 & -0.3 & -92108 & 173608 \\
w_2 & = & 0 & 0.3 & 0.00001 & 0.5 & -2 \\
w_3 & = & 0 & 92108 & -0.5 & 23840 & -44932 \\
w_4 & = & 0 & -173608 & 2 & -44932 & 84688 \\
\cline{3-7}
\end{array}$$

Since $m_{ij} = -m_{ji}$ for all $i \neq j$ except for $i = 4,\ j = 3$ and

$$\begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} 23840 & -44932 \\ -44932 & 84688 \end{bmatrix}$$

is positive definite, M is positive definite. After the six pivots $\langle w_4,z_4\rangle$, $\langle w_3,z_3\rangle$, $\langle w_2,z_2\rangle$, $\langle z_4,w_4\rangle$, $\langle z_3,w_3\rangle$ and $\langle z_2,w_2\rangle$, one returns to the initial schema.

Example 1 is a circling example in which M is of order four. Next we show that, in fact, four is the least dimension in which circling can occur.

Theorem 4. For circling to occur in the PPM applied to (q,M) where M is either a P-matrix or a positive semi-definite matrix, the order of M must be at least four.

Proof. Without loss of generality, we can assume that w_1 is the distinghished variable of the major cycle in which circling occurs. Let $K = \{i | i \neq 1,\ w_i$ becomes nonbasic during circling$\}$. In Section 2 and Section 3 it has been shown that, during circling, the algorithm merely chooses some $k \in K$ such that $m_{k1} < 0$ and then pivots on m_{kk} (if $m_{kk} > 0$) or on

$$\begin{bmatrix} m_{11} & m_{1k} \\ m_{k1} & m_{kk} \end{bmatrix} \quad (\text{if } m_{kk} = 0) .$$

If the order of M is less than four, then since $1 \notin K$, the cardinality of K is at most two. Since $\bar{m}_{k1}$, the pivotal transform of m_{k1}, is positive, $\bar{m}_{i1} < 0$ for at most one $i \in K$ during circling. Thus by default this negative $\bar{m}_{i1}$ has the smallest index among all $\bar{m}_{ji} < 0$ where $j \in K$. It follows that the least-index rule is implicitly imposed thereby making circling impossible, and the theorem follows. □

PART II

LEAST-INDEX RESOLUTION OF DEGENERACY IN

LEMKE'S ALGORITHM AND COTTLE'S ALGORITHM

1. Introduction.

Another, more robust, method for the LCP is due to Lemke [12], [14]. Since Lemke's algorithm is also a pivotal method, it is not surprising that it may circle when degeneracy occurs. Some circling examples will be given later.

In this Part, we impost a least-index rule on Lemke's algorithm and prove its finiteness when the matrix is either a P-matrix or a positive semi-definite matrix. We also show that Cottle's algorithm [3] for the parametric LCP is finite when the least-index rule is imposed.

2. Lemke's algorithm for (q,M).

2.1. A brief review of the algorithm.

Consider the auxiliary LCP

$$w = q + z_0 e + Mz \tag{1}$$

$$(w,\ z,\ z_0) \geq 0 \tag{2}$$

$$w^T z = 0 \tag{3}$$

where $e \in \mathbb{R}^n$, $e \geq 0$ and z_0 is an artificial variable. A solution of this system with $z_0 = 0$ is necessarily a solution of (q,M). If $e_i > 0$ for all i such that $q_i \leq 0$, then for $z = 0$ and z_0 suitably large, $w \geq 0$, and (2), (3) hold. Lemke's algorithm starts with such a z and z_0 and performs a sequence of pivots to achieve the condition $z_0 = 0$. Once $z_0 = 0$, a solution to (q,M) is obtained since during the process, (2) and (3) are always preserved.

Lemke's algorithm.

Step 0. Start with the basic solution $(w;z_0;z) = (q;0;0)$ and the matrix $\mathcal{M} = [e,M]$.

Step 1. If $q \geq 0$, stop. A solution $(w;z) = (q;0)$ is obtained. Otherwise, define k by $-q_k/e_k = \max_k\{-q_i/e_i\}$ and then pivot $\langle w_k, z_0\rangle$. Let $(\bar{q}, \bar{\mathcal{M}})$ denote the updated tableau and designate z_k, the complement of w_k, as the driving variable.

Step 2. If the driving variable z_k is unblocked, stop. Otherwise define j by

$$-\bar{q}_j/\bar{m}_{jk} = \min\{-\bar{q}_i/\bar{m}_{ik} \mid \bar{m}_{ik} < 0\} .$$

Step 3. If z_0 is the blocking variable, stop. A solution is at hand. Otherwise, perform the pivot $\langle w_j, z_k\rangle$ and let z_j, the complement of w_j, be the new driving variable. Return to Step 2 with the updated tableau.

In the nondegenerate case, Lemke's algorithm is finite [5], [12]. If z_k is unblocked in Step 2, we say that the algorithm terminates on

a secondary ray. When this happens, some results can be derived from the following theorem which is proved in [5].

Theorem 1. If Lemke's algorithm applied to (q,M) terminates on a ray, there exists a nonzero, nonnegative vector u such that

$$(4) \qquad u_i(Mu)_i \leq 0 \quad \text{for} \quad i = 1,2,\ldots, n.$$

In the case when M is a P-matrix, (4) cannot have a nonzero solution and consequently Lemke's algorithm will solve this problem. When M is positive semi-definite or, more generally, copositive-plus, termination on a ray implies that (q,M) is infeasible [12]. It is also well known that Lemke's algorithm can be applied to other classes of linear complementarity problems. Some detailed discussions can be found in [8], [9], and [13].

As mentioned above, Lemke's algorithm may circle when degeneracy occurs. Let us define the length of a circle to be the number of distinct basic sets in this circle. Kostreva [11] has shown that the minimum length of a circle in Lemke's algorithm for a general (q,M) is four. Moreover, he gives an example to illustrate that circling can occur when M is of order two.

Kosteva's circling example of order two, Example 2 of [11], uses an uncommon artifical vector e. Usually, e should be nonnegative (in order to initiate the process from a ray). If we always let the artificial vector e be nonnegative, his proof can still be applied to show that

the minimum length of a circle is four. But we will show that for circling to occur, M must be of order at least three. From the following example, it can be seen that Kostreva's bound on the length of circling is sharp.

Example 1. Apply Lemke's algorithm to (q,M) where

$$q = \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix} \qquad M = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Since $q_1 = 0$ and $q_2 < 0$, $q_3 < 0$, we let $e = (0,1,1)^T$. The initial schema is

	1	z_0	z_1	z_2	z_3
$w_1 =$	0	0	-1	-1	1
$w_2 =$	-2	1	1	1	0
$w_3 =$	-3	1	1	1	1

After the pivots $\langle w_3,z_0\rangle$, $\langle w_2,z_3\rangle$, we have the schema

(*)

	1	w_3	z_1	z_2	w_2
$w_1 =$	1	1	-1	-1	-1
$w_3 =$	1	1	0	0	-1
$z_0 =$	2	0	-1	-1	1

After further pivots $\langle w_1,z_2\rangle$, $\langle z_2,z_1\rangle$, $\langle z_1,w_2\rangle$ and $\langle w_2,w_1\rangle$, we return to (*) and a circle of length four is obtained.

Next we show that the matrix M in Example 1 has the least possible order for circling to occur.

<u>Proposition</u>. If M is of order two, Lemke's algorithm applied to (q,M) will terminate in a finite number of pivots for any q.

<u>Proof.</u> If $q \geq 0$, there is nothing to prove. Hence assume that $q_1 < 0$. Let $(e_1,e_2) = (1,e_2)^T$ where $e_2 \geq 0$ ($e_2 > 0$ if $q_2 < 0$).

$$
\begin{array}{r|c|c:cc|}
 & 1 & z_0 & z_1 & z_2 \\
\hline
w_1 = & q_1 & 1 & m_{11} & m_{12} \\
w_2 = & q_2 & e_2 & m_{21} & m_{22} \\
\hline
\end{array}
$$

Without loss of generality, we can assume that w_1 blocks z_0 when z_0 decreases to $-q_1$. Then the pivot $\langle w_1,z_0\rangle$ is performed. After the pivot, $\bar{e}_2 = e_2 \geq 0$

$$
\begin{array}{r|c|c:cc|}
 & 1 & w_1 & z_1 & z_2 \\
\hline
z_0 = & \bar{q}_1 & 1 & \bar{m}_{11} & \bar{m}_{12} \\
w_2 = & \bar{q}_2 & \bar{e}_2 & \bar{m}_{21} & \bar{m}_{22} \\
\hline
\end{array}
$$

Now z_1 is the new driving variable. If z_0 blocks z_1, terminate. Otherwise, w_2 blocks z_1 and the pivot $\langle w_2,z_1\rangle$ is performed. After the pivot, $\bar{\bar{e}}_2 \geq 0$ and z_2 is the new driving variable.

$$\begin{array}{r|c|c:cc|}
 & 1 & w_1 & w_2 & z_2 \\ \hline
z_0 = & \bar{\bar{q}}_1 & \bar{\bar{e}}_1 & \bar{\bar{m}}_{11} & \bar{\bar{m}}_{12} \\
z_1 = & \bar{\bar{q}}_2 & \bar{\bar{e}}_2 & \bar{\bar{m}}_{21} & \bar{\bar{m}}_{22} \\ \hline
\end{array}$$

If z_0 blocks z_2, terminate. Otherwise, z_1 is blocking and the pivot $\langle z_1, z_2 \rangle$ is performed.

$$\begin{array}{r|c|c:cc|}
 & 1 & w_1 & w_2 & z_1 \\ \hline
z_0 = & \hat{q}_1 & \hat{e}_1 & \hat{m}_{11} & \hat{m}_{12} \\
z_2 = & \hat{q}_2 & \hat{e}_2 & \hat{m}_{21} & \hat{m}_{22} \\ \hline
\end{array}$$

Now w_1 is the next driving variable. However, $\hat{e}_2 \geq 0$. Hence w_1 is either blocked by z_0 and a solution is obtained, or else w_1 is unblocked and the algorithm terminates on a ray. □

2.2. Lemke's algorithm with the least-index rule.

Even when M is a P-matrix, Lemke's algorithm applied to (q,M) may circle; e.g., see Example 1 of [11]. However, we show in this section that when M is a P-matrix or a positive semi-definite matrix, Lemke's algorithm with the least-index rule will process (q,M) in a finite number of steps. We say that the least-index rule is imposed in Lemke's algorithm

if, when there is a tie in choosing the exiting basic variable, we always choose the one with the least index as the exiting basic variable. Note that the index of the artificial variable z_0 is less than all other indices, hence it will be chosen if it is involved in a tie.

<u>Theorem 2</u>. When M is a P-matrix or a positive semi-definite matrix, Lemke's algorithm with the least-index rule will process (q,M) in a finite number of steps.

<u>Proof.</u> It is clear that when M is positive semi-definite, then so is the matrix $\mathcal{M}$ where

$$\mathcal{M} = \begin{bmatrix} \mu & e^T \\ -e & M \end{bmatrix}$$

for some real number $\mu \geq 0$. It is also clear that when M is a P-matrix, then so is the matrix $\mathcal{M}$ if μ is suitably large.

Lemke's algorithm starts with the system (1) where $z = 0$ and z_0 large enough so that $w \geq 0$, and then performs a sequence of pivots to achieve the condition $z_0 = 0$ (or else it goes off on a ray). Let $\bar{z}_0 > 0$ be the smallest value of z_0 such that $\bar{q} := q + \bar{z}_0 e \geq 0$. We can rewrite (1) as follows:

$$\begin{aligned} w &= q + z_0 e + Mz \\ &= \bar{q} + \alpha(-e) + Mz \end{aligned}$$

where

$$\alpha := \bar{z}_0 - z_0$$

Note that $\bar{q} \geq 0$ and α increases from 0 at the first step of Lemke's algorithm. Furthermore, $z_0 = 0$ if and only if $\alpha = \bar{z}_0$.

Therefore, by denoting

$$\gamma = \begin{pmatrix} q_0 \\ \bar{q} \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{bmatrix} \mu & e^T \\ -e & M \end{bmatrix}$$

where μ is sufficiently large and $q_0 < 0$ such that the absolute value of q_0 is sufficiently larger than μ, Lemke's algorithm on (q,M) can be regarded as a major cycle of the principal pivoting method on $(\gamma,\mathcal{M})$ in which α is the initial driving variable (α can be regarded as having the same index as z_0 does, namely 0).

$$\begin{array}{cc|c|cc|} & & 1 & \alpha & z \\ \cline{3-5} w_0 & = & q_0 & \mu & e^T \\ w & = & \bar{q} & -e & M \\ \cline{3-5} \end{array}$$

Since $z_0 = 0$ whenever $\alpha = \bar{z}_0$, in addition to the termination rules for the principal pivoting method, we terminate this major cycle when $\alpha = \bar{z}_0$ and in this case, a solution to (q,M) is obtained. However, it has been shown in Part I that each major cycle of the PPM with the least-index rule is finite, thus the result follows. □

Remark 1. From the above proof, it can also be seen that in Lemke's algorithm, the artificial variable z_0 is always monotonically decreasing when M is a P-matrix or a positive semi-definite matrix, Cottle [2].

Remark 2. Since the self-dual method [6] for linear programming is a special case of Lemke's algorithm, the least-index rule can be applied there also.

Corollary. The minimum length of a circle in the Lemke's algorithm without the least index rule applied to (q,M), where M is a P-matrix or a positive semi-definite matrix, is six. Furthermore, the bound is sharp.

Proof. The above proof shows that Lemke's algorithm can be regarded as part of a major cycle of PPM. Thus it follows from Theorem 4 of Part I that the minimum length of a circle is larger than or equal to six.

Example 1 of Kostreva [11] illustrates that six is the sharp bound for the P-matrix case. In the case of positive semi-definite matrix, let $\bar{M}$ be obtained from Example 1 of Part I by performing a pivot on m_{33}, then $\bar{M}_{i1} < 0$ for $i = 2,3,4$. Thus by considering $e = -(\bar{M}_{21}, \bar{M}_{31}, \bar{M}_{41})^T$, Example 1 of Part I and the proof for the above theorem show that the bound is sharp.

Theorem 2 shows that, when M is a P-matrix or a positive semi-definite matrix, Lemke's algorithm with the least-index rule will process the problem in a finite number of steps. Unfortunately, this is not true for a general matrix, even for a copositive-plus matrix. Example 2 below illustrates this.

Example 2. Consider the LCP (q,M) where

$$q = \begin{bmatrix} -10 \\ -10 \\ -10 \\ -8 \end{bmatrix}, \qquad M = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Clearly M is strictly copositive, and hence copositive - plus. Starting with the schema

	1	z_0	z_1	z_2	z_3	z_4
$w_1 =$	-10	1	2	3	3	2
$w_2 =$	-10	1	2	2	2	3
$w_3 =$	-10	1	2	3	3	1
$w_4 =$	-8	1	1	1	1	2

Lemke's algorithm with the least-index rule generates the pivoting sequence $\langle w_1,z_0\rangle$, $\langle w_4,z_1\rangle$, $\langle w_3,z_4\rangle$ and obtains the schema

(**)

	1	w_1	w_4	z_2	z_3	w_3
$z_0 =$	6	-3	2	1	1	2
$w_2 =$	0	2	0	-1	-1	-1
$z_4 =$	0	1	0	0	0	-1
$z_1 =$	2	1	-1	-2	-2	0

After the further pivots $\langle w_2,z_3\rangle$, $\langle z_3,z_2\rangle$, schema (**) becomes

$$
(***)\qquad
\begin{array}{r|c|ccccc|}
 & 1 & w_1 & w_4 & z_3 & w_2 & w_3\uparrow \\
\hline
z_0 = & 6 & -1 & 2 & 0 & -1 & 1 \\
z_2 = & 0 & 2 & 0 & -1 & -1 & -1 \\
z_4 = & 0 & 1 & 0 & 0 & 0 & -1 \\
z_1 = & 2 & -3 & -1 & 0 & 2 & 2 \\
\hline
\end{array}
$$

If the least-index rule is applied, then, after two more pivots $\langle z_2, w_3\rangle$, $\langle w_3, w_2\rangle$, schema (**) reappears and thus circling occurs.

However, if (instead of $\langle z_2, w_3\rangle$, $\langle w_3, w_2\rangle$) the pivot $\langle z_4, w_3\rangle$ is performed in (***), then after the further pivots $\langle z_1, w_4\rangle$, $\langle w_4, w_1\rangle$, $\langle w_3, z_4\rangle$, $\langle z_2, z_3\rangle$ and $\langle z_0, w_2\rangle$, a solution

$$w = (\tfrac{14}{5}, \tfrac{16}{5}, 0, 0), \qquad z = (0, 0, \tfrac{12}{5}, \tfrac{14}{5})$$

is obtained.

In this circling example, the artificial variable z_0 remains constant during circling. This is not true in general. For example, if $q_3 = -10$ is replaced by $q_3 = -9.5$ in Example 2, then Lemke's algorithm with the least-index rule generates the same circling sequence as above. However, during circling, the artificial variable z_0 no longer remains constant. The reason why the least-index rule works in P-matrix and positive semi-definite matrix cases but not in general remains to be found.

3. Cottle's algorithm for the parametric LCP.

By applying Grave's lexicographic principal pivoting method [10] for LCP, Cottle [3] developed a finite algorithm for checking the monotonicity of the solutions to the parametric LCP $\{(q + \alpha p, M) | \alpha \geq 0,\ q \geq 0\}$ in which M is a P-matrix or a positive semi-definite matrix. If the monotonicity check is deropped in this algorithm, we obtain an algorithm for solving the PLCP which we call Cottle's algorithm for PLCP. Without taking some (lexicographic) precautions for the constant column of [3], Cottle's algorithm may circle. For example, if $q_1 = -1$ is replaced by $q_1 = 1$ in Example 1 of Part I and $p = (1, 0.3, 92108, -173608)^T$, then circling occurs. However, we shall show here that Cottle's algorithm with the least-index rule is finite.

3.1. A brief review of the method.

Consider a PLCP $\{(q + \alpha p, M) | q \geq 0, \alpha \geq 0\}$ where M is either a P-matrix or a positive semi-definite matrix. Let Q be a matrix having linearly independent lexicographically positive rows and q as its first column $(q \neq 0)$. The lexicography is used as a cure for the degeneracy problem, not as a means to achieve greater generality.

Statement of Cottle's algorithm for PLCP

Step 0. Initialization. Start with α at the "critical value" $\bar{\alpha} = 0$ and set $z = 0$.

Step 1. If $p \geq 0$, stop. $(w;z) = (q + \alpha p;0)$ is a solution for all $\alpha \geq 0$. Otherwise

Step 2. Determine the critical index r by the condition

$-Q_r/P_r = \text{lexico}\ \min\{-Q_i/P_i | P_i < 0\}$. Set α equal to the

new critical value $\bar{\alpha} = q_r/-p_r$.

Step 3. Change of basis.

Case 1. $m_{rr} > 0$, then pivot on m_{rr} and return to Step 1 with transformed tableau.

Case 2. $m_{rr} = 0$.

If $M_{\cdot r} \geq 0$, stop. The problem is infeasible for all $\alpha > \bar{\alpha}$. Otherwise, define the index s by

$$-(Q_s - Q_r P_s/P_r)/m_{sr} = \text{lexico}\ \min\{-(Q_i - Q_r P_i/P_r)/m_{ir} | m_{ir} < 0\}$$

and pivot on

$$\begin{bmatrix} m_{rr} & m_{rs} \\ m_{sr} & m_{ss} \end{bmatrix} .$$

Return to Step 1 with the transformed tableau.

It can be shown that this algorithm is finite [3], [10]. This algorithm is also closely related to the Dantzig-Cottle principal pivoting method. In fact, we have the following:

Theorem 3. Suppose that M is a P-matrix or a positive semi-definite matrix. Let $\mu > 0$ be large enough such that

$$\mathcal{M} = \begin{bmatrix} \mu & -p^T \\ p & M \end{bmatrix}$$

is a P-matrix whenever M is. Assume $q_0 \ll 0$; then the execution of Cottle's PLCP algorithm is just a major cycle of the Dantzig-Cottle PPM on $(r, \mathcal{M})$ where

$$r = \begin{pmatrix} q_0 \\ q \end{pmatrix} .$$

<u>Proof</u>. It is clear that if M is positive semi-definite, then so is the matrix $\mathcal{M}$. Let (T1) be the initial schema of the PPM in which w_0 is the distinguished variable and z_0 is the driving variable. Let (T1*) be the initial schema of Cottle's PLCP algorithm.

(T1)

$$\begin{array}{r|c|cccccccccc|}
 & 1 & z_0 & z_1 & \cdots & z_r & \cdots & z_s & \cdots & z_n \\
\hline
w_0 = & q_0 & \mu & -p_1 & \cdots & -p_r & \cdots & p_s & \cdots & -p_n \\
w_1 = & q_1 & p_1 & m_{11} & \cdots & m_{1r} & \cdots & m_{1s} & \cdots & m_{1n} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
w_r = & q_r & p_r & m_{r1} & \cdots & m_{rr} & \cdots & m_{rs} & \cdots & m_{rn} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
w_s = & q_s & p_s & m_{s1} & \cdots & m_{sr} & \cdots & m_{ss} & \cdots & m_{sn} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
w_n & q_n & p_n & m_{n1} & \cdots & m_{nr} & \cdots & m_{ns} & \cdots & m_{nn} \\
\hline
\end{array}$$

$$
(\mathrm{T1}^*) \qquad
\begin{array}{r|c|cccccccc|}
 & 1 & \alpha\uparrow & z_1 & \cdots & z_r & \cdots & z_s & \cdots & z_n \\
\hline
w_1 = & q_1 & p_1 & m_{11} & \cdots & m_{1r} & \cdots & m_{1s} & \cdots & m_{1n} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
w_r = & q_r & p_r & m_{r1} & & m_{rr} & & m_{rs} & & m_{rn} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
w_s = & q_s & p_s & m_{s1} & \cdots & m_{sr} & \cdots & m_{ss} & \cdots & m_{sn} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
w_n = & q_n & p_n & m_{n1} & \cdots & m_{nr} & \cdots & m_{ns} & \cdots & m_{nn} \\
\hline
\end{array}
$$

If $p \geq 0$, then both PPM and Cottle's PLCP algorithm terminate at this step. Therefore, assume $p \not\geq 0$.

Suppose that r is the critical index in (T1*) and the critical value at this step is $\bar{\alpha} = q_r/-p_r$. Clearly w_r is also a blocking variable in (T1) when z_0 increases to $q_r/-p_r$.

If $m_{rr} > 0$, both PPM and Cottle's PLCP algorithm pivot on m_{rr}. After the pivot, the basic variables of PPM are identical to the basic variables of Cottle's PLCP algorithm except that PPM still has one more basic variable w_0. Furthermore, the common basic variables have the same values at this step, and the driving columns are still the same. It follows that the next blocking variables in these two algorithms will be identical.

Now suppose $m_{rr} = 0$. The PPM performs a block pivot of order 2 on the principal submatrix

$$
\begin{bmatrix} \mu & -p_r \\ p_r & 0 \end{bmatrix} .
$$

After the pivot, schema (T1) becomes the following schema (T2):

$$
\text{(T2)} \qquad
\begin{array}{rc|c|cccccccccc|}
 & & 1 & w_0 \uparrow & z_1 & \cdots & z_{r-1} & w_r & z_{r+1} & \cdots & z_s & \cdots & z_n \\
\hline
z_0 & = & \bar{q}_0 & \bar{\mu} & -\bar{p}_1 & \cdots & -\bar{p}_{r-1} & \bar{p}_r & \bar{p}_{r+1} & \cdots & \bar{p}_s & \cdots & \bar{p}_n \\
w_1 & = & \bar{q}_1 & \bar{p}_1 & & & & & & & & & \\
\vdots & & \vdots & \vdots & & & & & & & & & \\
w_{r-1} & = & \bar{q}_{r-1} & \bar{p}_{r-1} & & & & & & & & & \\
z_r & = & \bar{q}_r & \bar{p}_r & & & & & & & & & \\
w_{r+1} & = & \bar{q}_{r+1} & \bar{p}_{r+1} & & & & \bar{M} & & & & & \\
\vdots & & \vdots & \vdots & & & & & & & & & \\
w_s & = & \bar{q}_s & \bar{p}_s & & & & & & & & & \\
\vdots & & \vdots & \vdots & & & & & & & & & \\
w_n & = & \bar{q}_n & \bar{p}_n & & & & & & & & & \\
\hline
\end{array}
$$

where, for $i = 1,2,\ldots, n$ and $i \neq r$,

$$
(5) \qquad
\begin{cases}
\bar{p}_i = \dfrac{m_{ir}}{-p_r} \\[2ex]
\bar{q}_i = \dfrac{1}{p_r}\left(p_r q_i + m_{ir} q_0 - p_i q_r - \dfrac{m_{ir} q_r \mu}{p_r}\right).
\end{cases}
$$

Since w_r is the blocking variable in (T1), $p_r < 0$. Hence $\bar{p}$ has the same sign configuration as that of $M_{\cdot r}$. It follows that if $M_{\cdot r} \geq 0$, then both PPM and Cottle's PLCP algorithm terminate at this step.

Assume therefore that $M_{\cdot r} \not\geq 0$ and

$$(6) \qquad -(q_s - q_r p_s/p_r)/m_{sr} = \min\{-(q_i - q_r p_i/p_r)/m_{ir} \mid m_{ir} < 0\}$$

Then Cottle's PLCP algorithm pivots on

$$\begin{bmatrix} 0 & m_{rs} \\ m_{sr} & m_{ss} \end{bmatrix} .$$

After the pivot, (T1*) becomes

$$(\mathrm{T2^*}) \qquad \begin{array}{rc|c|c}
 & & 1 & \alpha \quad z_1 \ \cdots \ z_{r-1} \ w_r \ z_{r+1} \ \cdots \ z_{s-1} \ w_s \ z_{s+1} \ \cdots \ z_n \\
\hline
w_1 & = & \tilde{q}_1 & \tilde{p}_1 \\
\vdots & & \vdots & \vdots \\
w_{r-1} & = & \tilde{q}_{r-1} & \tilde{p}_{r-1} \\
z_r & = & \tilde{q}_r & \tilde{p}_r \\
w_{r+1} & = & \tilde{q}_{r+1} & \tilde{p}_{r+1} \\
\vdots & & \vdots & \vdots \qquad\qquad\qquad \tilde{M} \\
w_{s-1} & = & \tilde{q}_{s-1} & \tilde{p}_{s-1} \\
z_s & = & \tilde{q}_s & \tilde{p}_s \\
w_{s+1} & = & \tilde{q}_{s+1} & \tilde{p}_{s+1} \\
\vdots & & \vdots & \vdots \\
w_n & = & \tilde{q}_n & \tilde{p}_n \\
\hline
\end{array}$$

Note that (6) implies that in (T1), (T1*), when

$$z_0 = \alpha = q_r/-p_r ,$$

$$z_r = -(q_s - q_r p_s/p_r)/m_{sr} ,$$

and

$$z_i = 0 , \qquad \text{for all } i \neq r, 0 ,$$

we have

$$
(7)\quad \begin{cases} w_0 = \bar{w}_0 := q_0 - \dfrac{q_r}{p_r}\mu + \dfrac{p_r}{m_{sr}}\left(q_s - \dfrac{q_r p_s}{p_r}\right), \\ w_s = 0, \\ w_i \geq 0, \quad \text{for all} \ \ i \neq s, 0 \ . \end{cases}
$$

Now, consider the schema (T2) in which $z_1 = \cdots = z_{r-1} = w_r = z_{r+1} = \cdots = z_n = 0$. When w_0 increases to $\bar{w}_0$, then by (5) and (7):

$$
\begin{aligned}
w_s &= \bar{q}_s + \bar{p}_s \bar{w}_0 \\
&= \frac{1}{p_r}\left(p_r q_s + m_{sr} q_0 - p_s q_r - \frac{m_{sr} q_r \mu}{p_r}\right) \\
&\quad - \frac{m_{sr}}{p_r}\left[q_0 - \frac{q_r}{p_r}\mu + \frac{p_r}{m_{sr}}\left(q_s - \frac{q_r p_s}{p_r}\right)\right] \\
&= 0 \ .
\end{aligned}
$$

It can also be easily checked that $w \geq 0$ at this step. Therefore, w_s is a blocking variable in (T2). In schema (T2), we have

$$
\bar{m}_{ss} = m_{ss} + \frac{m_{sr}^2}{p_r^2} \quad .
$$

It follows from (6) that $m_{sr} \neq 0$ and hence $\bar{m}_{ss} > 0$. Therefore, the PPM performs a pivot on $\bar{m}_{ss}$, and after the pivot (T2) becomes

$$
(T3)\qquad
\begin{array}{rc|c|cccccccccccc|}
 & & 1 & w_0 & z_1 & \cdots & z_{s-1} & w_s & z_{s+1} & \cdots & z_{r-1} & w_r & z_{r+1} & \cdots & z_n \\
\hline
z_0 & = & \bar{\bar{q}}_0 & \bar{\bar{\mu}} & -\bar{\bar{p}}_1 & \cdots & -\bar{\bar{p}}_{s-1} & -\bar{\bar{p}}_s & -\bar{\bar{p}}_{s+1} & \cdots & -\bar{\bar{p}}_{r-1} & -\bar{\bar{p}}_r & -\bar{\bar{p}}_{r+1} & \cdots & -\bar{\bar{p}}_n \\
w_1 & = & \bar{\bar{q}}_1 & \bar{\bar{p}}_1 & & & & & & & & & & & \\
\vdots & & \vdots & \vdots & & & & & & & & & & & \\
w_{s-1} & = & \bar{\bar{q}}_{s-1} & \bar{\bar{p}}_{s-1} & & & & & & & & & & & \\
z_s & = & \bar{\bar{q}}_s & \bar{\bar{p}}_s & & & & & & & & & & & \\
w_{s+1} & = & \bar{\bar{q}}_{s+1} & \bar{\bar{p}}_{s+1} & & & & & & & & & & & \\
\vdots & & \vdots & \vdots & & & & & & \bar{\bar{M}} & & & & & \\
w_{r-1} & = & \bar{\bar{q}}_{r-1} & \bar{\bar{p}}_{r-1} & & & & & & & & & & & \\
z_r & = & \bar{\bar{q}}_r & \bar{\bar{p}}_r & & & & & & & & & & & \\
w_{r+1} & = & \bar{\bar{q}}_{r+1} & \bar{\bar{p}}_{r+1} & & & & & & & & & & & \\
\vdots & & \vdots & \vdots & & & & & & & & & & & \\
w_n & = & \bar{\bar{q}}_n & \bar{\bar{p}}_n & & & & & & & & & & & \\
\hline
\end{array}
$$

(T2*) and (T3) have the same set of basic variables except that z_0 is a basic variable in (T3) and α is a parameter in (T2*). Hence, (T2*) can be regarded as a subschema obtained by performing a pivot on $\bar{\bar{\mu}}$ in (T3). It follows that $\tilde{p} = \bar{\bar{p}}/\bar{\bar{\mu}}$ and the corresponding variables of (T2*) and (T3) have the same values. Therefore the next blocking variables in the PPM and Cottle's PLCP algorithm will be identical.

By repeating the above argument, it follows that the execution of Cottle's PLCP algorithm is just a major cycle of the PPM on (r, M). □

3.2. Cottle's PLCP algorithm with the least-index rule.

In this section, we shall give a modification of Cottle's algorithm and prove its finiteness without recourse to a lexicopositive matrix Q. To accomplish this, we introduce a refinement of Cottle's algorithm which imposes the following least-index rule:

(i) In Step 2, determine the critical index r by

$$r = \min_k \{k \mid p_k \leqslant 0, \ -q_k/p_k = \min\{-q_i/p_i \mid p_i < 0\}\}$$

(ii) In the Case 2 of Step 3, define s by

$$s = \min\{k \mid m_{kr} < 0, \ -(q_k - q_r p_k/p_r)/m_{kr} = \min\{-(q_i - q_i p_i/p_r)/m_{ir} \mid m_{ir} < 0\}\}$$

In (i) and (ii), $M_{\cdot k}$ represents the column of M corresponding to the current nonbasic variable z_k. Similarly, $(q_k, p_k, M_{k\cdot})$ represents the row of the schema corresponding to the current basic variable w_k.

<u>Theorem 4</u>. Cottle's PLCP algorithm with the least-index rule will terminate in a finite number of steps.

<u>Proof.</u> Theorem 3 shows that the execution of Cottle's PLCP algorithm is just a major cycle of the Dantzig-Cottle PPM. Therefore, the result follows from Theorem 2 and Theorem 3 of Part 1. □

4. An extension of Murty's scheme for (q,M).

Murty's scheme [15] can be applied to (q,M) only when M is a P-matrix. In this section we give an extension of Murty's scheme to solve (q,M) in which M is a positive semi-definite matrix. In the following, M is either a P-matrix or a positive semi-definite matrix.

Statement of a direct scheme for (q,M)

Step 0. Set $h = 0$. Begin with the system $w^h = q^h + M^h z^h$ where $w^o = q^0 + M^0 z^0$ is the given system $w = q + Mz$. The nonbasic vector z^h always assumes value 0.

Step 1. If $q^h \geq 0$, stop. $[w^h; z^h] = [q^h; 0]$ is a solution. Otherwise, choose $\gamma = \min\{i \mid w_i^h < 0\}$.

Step 2. Change of basis.

Case 1. $m_{rr}^h > 0$, then pivot on m_{rr}^h. Set $h = h+1$ and return to Step 1.

Case 2. $m_{rr}^h = 0$.

If $M_{\cdot r}^h \geq 0$, stop. The problem is infeasible.

Otherwise, choose

$$s = \min\{i \mid m_{ir}^h < 0\}$$

and pivot on

$$\begin{bmatrix} m_{rr}^h & m_{rs}^h \\ m_{sr}^h & m_{ss}^h \end{bmatrix}$$

Set $h = h+1$ and return to Step 1.

Theorem 5. When M is either a P-matrix or a positive semi-definite matrix, the above scheme will process (q,M) in a finite number of pivots.

Proof. The above scheme generates the same pivoting sequence as that generated by Cottle's algorithm with the least-index rule for the PLCP $\{(0 + \alpha q,M)|\alpha \geq 0\}$ where 0 is a zero vector. Therefore, the result follows from Theorem 4. $\square$

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TECHNICAL REPORT 79-14 Yow-Yieh Change

LEAST-INDEX RESOLUTION OF DEGENERACY IN LINEAR COMPLEMENTARITY PROBLEMS

This study center on the circling phenomenon associated with degeneracy in linear complementarity problems and presents an easily implemented technique for resolving it. With certain exceptions, the device is to use the least-index for selecting the variable to leave the basic set.

The results of this report pertain only to linear complementarity problems involving P-matrices or positive semi-definite matrices. With this restriction, it is shown that inclusion of the least-index pivot selection rule insures finiteness for the principal pivoting method of Dantzig and Cottle, Lemke's algorithm, and Cottle's parametric principal pivoting method. It is shown that for circling to occur in the principal pivoting method, the matrix must have order at least four, and for Lemke's algorithm it must be at least three. Examples are given showing that these bounds are sharp. Finally, Murty's version of Bard's method is extended from P-matrices to the positive semi-definite case.

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